

FUNCTORIALITY OF COULOMB BRANCHES - TALK NOTES

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ABSTRACT. These are combined notes from a talk I gave at the *Categorification and Symplectic Duality Workshop* at Northeastern and from a talk I gave at the *Informal String-Math Seminar* at UC Berkeley, both in 2025.

0.1. Motivation from Physics and Symplectic Duality. Our reference here is [BF18, Section 4]. To any 3d $\mathcal{N} = 4$ supersymmetric gauge theory $\mathcal{J}$, there is supposed to be a mathematically defined object called its moduli space of vacua. As far as I understand, this space is considered too complicated to be given a precise mathematical footing for now. However, there are two easier ‘parts’ of this space which mathematicians have a hope of understanding: namely, the *Higgs branch* and the *Coulomb branch* of $\mathcal{J}$. One case where a lot of interesting mathematics has been done is the case where $\mathcal{J}$ is a *gauge theory of cotangent type*—in other words, to any complex reductive group G and any finite dimensional complex representation, there should be an associated 3d $\mathcal{N} = 4$ supersymmetric gauge theory $\mathcal{J}(G, \mathbf{N})$.¹ The Higgs and Coulomb branches of this space (to be defined later) have played a central role in a field known as *symplectic duality*. In fact, it’s played such a relevant role that we will take the following quote as our ‘good first approximation’ to symplectic duality:

‘To any complex reductive group G and any finite dimensional representation $\mathbf{N}$, there is family of interesting conjectures which relate the *Higgs branch* $\mathcal{M}_H(G, \mathbf{N})$ to the *Coulomb branch* $\mathcal{M}_C(G, \mathbf{N})$.’

Today, we’ll first discuss the work that’s been done to make the easy pieces of this space, i.e. the Higgs branch and the Coulomb branch, precise.

0.2. The Higgs Branch and Symplectic Duality.

0.2.1. Higgs Branch. Let’s first define what we mean by the word Higgs branch. To do this, let me first recall the following general fact:

Proposition 0.1. If G acts on any scheme X , then there is a natural G -action on the *cotangent bundle* $T^*(X)$ and a G -equivariant map $\mu : T^*(X) \rightarrow \mathfrak{g}^*$.

The map μ is called the *moment map* for the G -action on $T^*(X)$.

Proof of Proposition 0.1. First, recall that if G acts on any scheme X , then any tangent vector at the identity $\xi \in \mathfrak{g} := T_1(G)$ gives rise to a *vector field* on X . To see this, observe that we have an action map

$$\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \times X \xrightarrow{\xi \times \mathrm{id}} G \times X \rightarrow X$$

compatible with the projection onto X —the compatibility with the projection map follows from the fact that ξ is a tangent vector at the *identity* of G . Let $\tilde{\xi}$ denote this vector field. Our map is informally given by the formula $(x, \nu \in T_x^*(X)) \mapsto (\xi \mapsto \nu(\tilde{\xi}|_x))$, which I also claim is G -equivariant. $\square$

Definition 0.2. The *Higgs branch* $\mathcal{M}_H(G, \mathbf{N})$ associated to the G -representation $\mathbf{N}$ is the affine scheme

$$\mathcal{M}_H(G, \mathbf{N}) := \mu^{-1}(0) // G := \mathrm{Spec}(\mathrm{Fun}(T^*(\mathbf{N}))/\mathrm{Fun}(T^*(\mathbf{N}))\mathfrak{g})^G$$

sometimes known as the *Hamiltonian reduction* of $T^*(\mathbf{N})$ by its (Hamiltonian) G -action.

¹In fact, this theory should only depend on the representation $T^*\mathbf{N}$ of the *cotangent bundle* of $\mathbf{N}$.

Example 0.3. Let $G := \mathbb{G}_m$; the formula

$$\alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$$

gives a G -representation $\mathbf{N} := \mathbb{A}^2 = \text{Spec}(\mathbb{C}[x_1, x_2])$. Any G -action on an affine scheme is determined by a grading on its ring of functions and this action gives x_1, x_2 grading 1. Then $T^*(\mathbf{N}) = \text{Spec}(\mathbb{C}[x_1, x_2, \xi_1, \xi_2])$ has a G -action where ξ_1, ξ_2 have grading -1, and

$$\begin{aligned} \mathcal{M}_H(G, \mathbb{A}^2) &= \text{Spec}(\mathbb{C}[x_1, x_2, \xi_1, \xi_2]/(x_1\xi_1 + x_2\xi_2))^G = \text{Spec}(\mathbb{C}[a := x_1\xi_1, b := x_1\xi_2, c := x_2\xi_1, d := x_2\xi_2]/(a+d, ad-bc)) \\ &= \text{Spec}(\mathbb{C}[a, b, c]/(-a^2 - bc)) = \{M := \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : \det(M) = 0\} \end{aligned}$$

is the *nilpotent cone* of $\mathfrak{sl}_2$.

This example generalizes where $\mathcal{A}^2$ is replaced with the scaling action on $\mathcal{A}^n$. In this case, it turns out that $\mathcal{M}_H(G, \mathcal{A}^n) = \{M \in \mathcal{A}^{n \times n} : M^2 = 0, \text{rank}(M) \leq 1\}$.

0.2.2. Symplectic Duality: The (Overgeneralized) Hikita Conjecture. The *Coulomb branch* $\mathcal{M}_C(G, \mathbf{N})$ associated to a G -representation $\mathbf{N}$ is a certain affine variety whose smooth locus admits a symplectic form. Its definition more seriously draws on ideas in string theory, and is more subtle than that of the Higgs branch. Before defining it more precisely, let's see one of the 'symplectic duality' conjectures that one can test, an overgeneralization of something called the *Hikita conjecture*:

Conjecture 0.4. Assume $\mathcal{M}_H(G, \mathbf{N})$ admits a *symplectic resolution*, that is, a resolution of singularities $\widetilde{\mathcal{M}}_H(G, \mathbf{N}) \rightarrow \mathcal{M}_H(G, \mathbf{N})$ for which the symplectic form on the smooth locus of $\mathcal{M}_H(G, \mathbf{N})$ extends to a symplectic form on $\widetilde{\mathcal{M}}_H(G, \mathbf{N})$. Then there is a $\mathbb{G}_m$ -action on $\mathcal{M}_C(G, \mathbf{N})$ for which $\mathcal{M}_C(G, \mathbf{N})^{\mathbb{G}_m} = \text{Spec}(H^*(\widetilde{\mathcal{M}}_H(G, \mathbf{N}), \mathbb{C}))$.

Example 0.5. Before Coulomb branches were even defined mathematically precisely, it was known that, in the example of $G := \mathbb{G}_m$ scaling $\mathbf{N} := \mathbb{A}^2$ as above, that the Coulomb branch $\mathcal{M}_C(G, \mathbf{N})$ just so happens to *also* be the nilpotent cone for $\mathfrak{sl}_2$. If you're a representation theorist, then, you might know that $\mathcal{M}_C(G, \mathbf{N})$ admits a resolution called the *Springer resolution*

$$\widetilde{\mathcal{M}}_C(G, \mathbf{N}) := T^*(\mathbb{P}^1) \rightarrow \mathcal{M}_C(G, \mathbf{N})$$

which turns out to be symplectic where the symplectic form on $T^*(\mathbb{P}^1)$ is the symplectic form which comes by nature of it being a cotangent bundle. Observe that

$$H^*(T^*(\mathbb{P}^1), \mathbb{C}) = H^*(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}[\epsilon]/\epsilon^2$$

and, if we equip the ring $A := \mathbb{C}[a, b, c]/(-a^2 - bc)$ with a grading so that a has degree 0, b has degree 2, and c has degree -2 (or if b has degree 1 and c has degree -1) we see that

$$\text{Spec}(A)^{\mathbb{G}_m} = \text{Spec}(A_0/A_2A_{-2}) = \text{Spec}(\mathbb{C}[a])/a^2$$

and so these rings match. Thus Conjecture 0.4 is true for $G = \mathbb{G}_m$ and $\mathbf{N} = \mathcal{A}^2$.

0.3. Introduction to Coulomb Branches.

0.3.1. $\mathcal{M}_C(\text{GL}_2, 0)$. Let's discuss the geometry of one of the simplest cases where G is not abelian: the Coulomb branch $\mathcal{M}_C(\text{GL}_2, 0)$. There will be a similar description for $\mathcal{M}_C(G, 0)$, and, even if you don't know what a Coulomb branch is, it turns out that $\mathcal{M}_C(G, 0)$ completely determines $\mathcal{M}_C(G, \mathbf{N})$ in a way that can be made explicit in most cases. In fact, we will appeal immediately to a result of [BFM05] which says that, whatever the definition of $\mathcal{M}_C(\text{GL}_2, 0)$ is, it's isomorphic to something called the *group scheme of universal centralizers* for GL_2 . This means that, by construction, there is a map

$$\mathcal{M}_C(\text{GL}_2, 0) \rightarrow \mathfrak{gl}_2 // \text{GL}_2 \cong \mathbb{A}^2 // S_2$$

where the fiber over any $\{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$ is $\mathbb{G}_m \times \mathbb{G}_m$ and the fiber over any $\{\lambda, \lambda\}$ is $\mathbb{G}_m \times \mathbb{G}_a$. Another way to see this is that if $\{\lambda_1, \lambda_2\}$ are the unordered roots to the polynomial $p(t) = t^2 + c_1t + c_0$, then the fiber $\mathcal{M}_C(\text{GL}_2, 0) \times_{\mathbb{A}^2 // S_2} \{\lambda_1, \lambda_2\}$ is the centralizer of the *companion matrix* $\begin{pmatrix} 0 & -c_0 \\ 1 & -c_1 \end{pmatrix}$. Now, this is an informal description of the Coulomb branch $\mathcal{M}_C(\text{GL}_2, 0)$: it an affine algebraic variety that lives over

$\mathfrak{gl}_2 // \mathrm{GL}_2 \cong \mathcal{A}^2 // S_2 \cong \mathbb{A}^2$ whose fiber is typically $\mathbb{G}_m \times \mathbb{G}_m$ but at a small set of points the fiber is different. There is an obvious generalization of this to GL_n .

Observe that $\mathcal{M}_C(\mathrm{GL}_2, 0)$ is a group *scheme* over $\mathfrak{gl}_2 // \mathrm{GL}_2$. More precisely, we can multiply two elements of $\mathcal{M}_C(\mathrm{GL}_2, 0)$ and get an element of $\mathcal{M}_C(\mathrm{GL}_2, 0)$ as long as the two elements have the same image under the map $\mathcal{M}_C(\mathrm{GL}_2, 0)$. It will turn out that $\mathcal{M}_C(G, 0)$ is always a group scheme over $\mathfrak{g} // G$, but this won't be true for $\mathcal{M}_C(G, \mathbf{N})$.

0.3.2. Properties of Coulomb Branches. In the seminal work [BFN18], the authors propose the definition of a *Coulomb branch* $\mathcal{M}_C(G, \mathbf{N})$ associated to a complex reductive group G and finite dimensional representation $\mathbf{N}$. From my point of view, it helps to first see what a variety being a Coulomb branch buys you rather than its formal definition:

Theorem 0.6. The scheme $\mathcal{M}_C(G, \mathbf{N})$ has the following properties:

- (1) [BFN18] $\mathcal{M}_C(G, \mathbf{N})$ is a normal affine variety *variety*, in other words it is the Zariski spectrum of an integral domain that is finitely generated as a $\mathbb{C}$ -algebra integrally closed in its field of fractions
- (2) [BFN18] There is a flat map $\varpi_{\mathbf{N}} : \mathcal{M}_C(G, \mathbf{N}) \rightarrow \mathfrak{g} // G$ whose fibers are generically isomorphic to $\mathbb{G}_m^n$. Moreover, there is a map $\mathcal{M}_C(G, 0) \rightarrow \mathcal{M}_C(G, \mathbf{N})$ naturally compatible the maps to $\mathfrak{g} // G$ in the natural way, and there is an open subset of $\mathfrak{g} // G$ where this map is an isomorphism.
- (3) [BFN18], [Wee22], [Bel23] The smooth locus of $\mathcal{M}_C(G, \mathbf{N})$ admits a symplectic form. Moreover, $\mathcal{M}_C(G, \mathbf{N})$ has *symplectic singularities* in the sense of [Bea99] and thus ([Kal06]) admits a finite stratification by its *symplectic leaves*
- (4) [Tel23] There is an action of $\mathcal{M}_C(G, 0)$ on $\mathcal{M}_C(G, \mathbf{N})$.

0.3.3. The Convolution Product and Coulomb Branches. To motivate the definition of Coulomb branches in general, first let me remind you that the homology of any Lie group G admits an algebra structure: the multiplication is

$$H_*(G) \otimes H_*(G) \cong H_*(G \times G) \xrightarrow{H_*(\mathrm{mult})} H_*(G).$$

Given a subgroup $P \subseteq G$, we can define a multiplication on the P -equivariant homology of G/P . It's easiest to informally identify $H_*^P(G/P) \cong H_*(P \backslash G/P) \cong H_*^{P \times P}(G)$

$$H_*(P \backslash G/P) \otimes H_*(P \backslash G/P) \cong H_*(P \backslash G/P \times P \backslash G/P) \xrightarrow{\mathrm{pull\ back}} H_*(P \backslash G \times^P G/P) \xrightarrow{H_*(\mathrm{mult})} H_*(P \backslash G/P)$$

which we can do since the natural projection map is smooth and the multiplication map is proper.

Let's apply this more general framework to the *affine Grassmannian* $\mathrm{Gr}_G := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. We therefore obtain a multiplication map

$$(1) \quad H_*^{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G) \otimes_{H_*^{G(\mathbb{C}[[t]])}(\mathrm{pt})} H_*^{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G) \cong H_*^P(G \times G/P) \xrightarrow{H_*(\mathrm{act})} H_*^P(G/P).$$

Theorem 0.7. [BFM05]

- (1) The algebra $H_*^{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ is commutative(!).
- (2) The commutative algebra $H_*^{\mathrm{GL}_n(\mathbb{C}[[t]])}(\mathrm{Gr}_{\mathrm{GL}_n})$ gives the ring of algebraic functions on $\mathcal{M}_C(\mathrm{GL}_n, 0)$ discussed above.
- (3) More generally, the commutative algebra $H_*^{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ gives the ring of algebraic functions on the *group scheme of universal centralizers* $J_{G^\vee}$ for the **Langlands dual group** $G^\vee$.

Now we define Coulomb branches. To any $(G, \mathbf{N})$ one can construct a certain equivariant map $\pi : \mathcal{R}_{G, \mathbf{N}} \rightarrow \mathrm{Gr}_G$ whose fiber at any point $[g] \in \mathrm{Gr}_G$ is $\mathbf{N}(\mathbb{C}[[t]]) \cap g \cdot \mathbf{N}(\mathbb{C}[[t]])$. The insight of [BFN18] is that one can equip the equivariant Borel-Moore homology $H_{\mathrm{BM}}^{G(\mathbb{C}[[t]])}(\mathcal{R}_{G, \mathbf{N}})$ with a similar-in-spirit convolution monoidal structure. However, much more care and work needs to go into defining this convolution monoidal structure—you can check that, for example, over the basepoint $1 \in \mathrm{Gr}_G$ that the fiber is $\mathbf{N}(\mathbb{C}[[t]])$ —an infinite dimensional vector space. This is in stark contrast to Gr_G , which one can show is an increasing union of finite type schemes and thus, informally speaking, is easier to work with.

Theorem 0.8. [BFN19] The convolution monoidal structure on $H_{\mathrm{BM}}^{G(\mathbb{C}[[t]])}(\mathcal{R}_{G, \mathbf{N}})$ is commutative.

Definition 0.9. [BFN19] The *Coulomb branch* associated to $(G, \mathbf{N})$ is the affine scheme whose ring of functions is the commutative algebra $H_{\mathrm{BM}}^{G(\mathbb{C}[[t]])}(\mathcal{R}_{G, \mathbf{N}})$.

0.3.4. *Alternate Definition for Gluable Representations.* Let $\varphi_1, \dots, \varphi_d$ be the weights of the representation $\mathbf{N}$, considered as a multiset of linear functions on $\mathfrak{t}$. Each such weight can also be considered as a cocharacter $\mathbb{G}_m \rightarrow T^\vee$ to the dual torus, which we denote $a \mapsto a^{\varphi_i}$. Thus, any non-zero weight defines a rational map $\mathfrak{t} \dashrightarrow T^\vee$ defined by $\zeta \mapsto \varphi_i(\zeta)^{\varphi_i}$. The product of these functions defines a rational map

$$\epsilon_{\mathbf{N}} : \mathfrak{t} \dashrightarrow T^\vee \quad \epsilon_{\mathbf{N}}(\zeta) = \prod_{i=1}^d \varphi_i(\zeta)^{\varphi_i}.$$

Since $\mathbf{N}$ is a $\tilde{G}$ -representation, the multi-set $\{\varphi_1, \dots, \varphi_d\}$ is W -invariant, and so $\epsilon_{\mathbf{N}}$ is equivariant. The functoriality of Borel-Moore homology gives a birational map

$$\mathcal{M}_C(G)_{\mathfrak{t}} \rightarrow \mathcal{M}_C(T)$$

(say, by [BFN18, Lemma 5.17]) and so $\epsilon_{\mathbf{N}}$ defines a W -equivariant rational map

$$(2) \quad \mathcal{M}_C(G)_{\mathfrak{t}} \rightarrow \mathcal{M}_C(T) = T^\vee \times \mathfrak{t} \dashrightarrow T^\vee \times \mathfrak{t} = \mathcal{M}_C(T) \leftarrow \mathcal{M}_C(G)_{\mathfrak{t}}$$

where the dashed right arrow is given by the formula $(x, \zeta) \mapsto (x\epsilon_{\mathbf{N}}(\zeta), \zeta)$. Taking the GIT quotient by the Weyl group W , we obtain a rational map

$$(3) \quad \mathcal{M}_C(G, 0) \dashrightarrow \mathcal{M}_C(G, 0)$$

or, in other words, an injection on the corresponding field of functions of $\mathcal{M}_C(G, 0)$.

Definition 0.10. We will say that a representation $\mathbf{N}$ of a complex reductive group G is *gluable* if there are no nonzero weights ξ_1, ξ_2 of $\mathbf{N}$ such that $\xi_1 = -\alpha\xi_2$ for some *positive* $\alpha \in \mathbb{Q}$.

The following result is largely due to Teleman; we give a slight weakening of the hypotheses:

Theorem 0.11. [Tel21] [GW25] The Coulomb branch is the affine scheme whose ring of functions are those of $\mathcal{M}_C(G, 0)$ which remain regular after pulling back by the map (3).

Remark 0.12. One can more algebro-geometrically state Theorem 0.11 as saying that, for gluable representations $\mathbf{N}$, the scheme $\mathcal{M}_C(G, \mathbf{N})$ can be written as a coproduct $\mathcal{M}_C(G, 0) \coprod_{\mathcal{M}_C(G, 0)^\circ} \mathcal{M}_C(G, 0)$ in the category of affine schemes, where $\mathcal{M}_C(G, 0)^\circ$ is a certain explicit open subset of $\mathcal{M}_C(G, 0)^\circ$ and one of the inclusion maps is twisted by (3). This motivates the term ‘gluable.’ It also gives a concrete explanation for why $\mathcal{M}_C(G, 0)$ acts on $\mathcal{M}_C(G, \mathbf{N})$ —the maps in this coproduct are equivariant with respect to the $\mathcal{M}_C(G, 0)$ -action!

0.3.5. *Functoriality of Coulomb Branches.* After the definition of Coulomb branches was made precise, the following philosophy started to emerge:

‘To any map $H \rightarrow G$ of complex reductive groups and any representation $\mathbf{N}$ of G , the Coulomb branch $\mathcal{M}_C(G, \mathbf{N})$ completely determines $\mathcal{M}_C(H, \mathbf{N})$.

For example, this philosophy is used to defined Coulomb branches of *non-cotangent type*, see [Bra+22], [Tel23]. Our main result in this paper is a particular instantiation of this philosophy that lends itself well to computations, which we will discuss after stating the main theorem. To start, let’s just fix the data of a map $H \rightarrow G$ and a representation $\mathbf{N}$ of G . First, observe that there is a natural map of affine Grassmannians $\mathrm{Gr}_H \rightarrow \mathrm{Gr}_G$. Therefore, we obtain an induced map on equivariant Borel-Moore homology

$$H_*^H(\mathrm{Gr}_H) \rightarrow H_*^H(\mathrm{Gr}_G) \cong H_H^*(\mathrm{pt}) \otimes_{H_G^*(\mathrm{pt})} H_G^*(\mathrm{Gr}_G)$$

and so we obtain an induced map

$$(4) \quad \mathcal{M}_C(G, 0) \times_{\mathfrak{g} // G} \mathfrak{h} // H \rightarrow \mathcal{M}_C(H, 0)$$

of group schemes over $\mathfrak{h} // H$. I haven’t told you the full legal definition of $\mathcal{R}_{G, \mathbf{N}}$ but I claim that there is a similar induced map

$$(5) \quad \mathcal{M}_C(G, \mathbf{N}) \times_{\mathfrak{g} // G} \mathfrak{h} // H \rightarrow \mathcal{M}_C(H, \mathbf{N})$$

of affine schemes. Now here is the key observation: both the domain and codomain of (5) have actions of $\mathcal{M}_C(G, 0) \times_{\mathfrak{g} // G} \mathfrak{h} // H$ (the latter from using the map (4)) and so we get an induced map

$$(6) \quad \mathcal{M}_C(H, 0) \times_{\mathfrak{g} // G}^{\mathcal{M}_C(G, 0)} \mathcal{M}_C(G, \mathbf{N}) \cong \mathcal{M}_C(H, 0) \times_{\mathfrak{h} // H}^{\mathcal{M}_C(G, 0) \times_{\mathfrak{g} // G} \mathfrak{h} // H} (\mathcal{M}_C(G, \mathbf{N}) \times_{\mathfrak{g} // G} \mathfrak{h} // H) \rightarrow \mathcal{M}_C(H, \mathbf{N})$$

of affine schemes.

Theorem 0.13. Assume the map $H \rightarrow G$ is gluable.² Then the map (6) is an isomorphism of affine schemes.

We'll soon discuss particular instantiations of this, so that you see the gluing hypothesis is not vacuous. One particular way that we use this is to prove that certain varieties which arise in the context of geometric representation theory are Coulomb branches.

Remark 0.14. The same arguments used to prove Theorem 0.13 also prove the variant for *K-theoretic Coulomb branches*.

0.4. The Base Affine Space. Many recent results of mine [GW23], [GW25], are that certain varieties appearing naturally in geometric representation theory are Coulomb branches. Recall that irreducible representations of a complex reductive group G are classified by a *highest weight vector*, that is, a vector fixed by a maximal unipotent subgroup $U \subseteq G$. Therefore, acting on a highest weight vector in an irreducible representation induces a map from G/U .

Definition 0.15. The *base affine space* of a reductive group G is the quotient G/U .

Experience in representation theory suggests that, for many problems in representation theory, the solution results from a careful study of the base affine space. In particular, the structure of the ring of differential operators on G/U seems to be closely connected with the representations of the real forms of G .

– J. Bernstein, S. Gelfand, and I. M. Gelfand [BGG75]

Example 0.16. $\mathrm{SL}_2/U \cong \{(x, y) : (x, y) \neq (0, 0)\}$

Example 0.17. $\mathrm{SL}_3/U \cong \{(a, b, c, x, y, z) : ax + by + cz = 0, (a, b, c) \neq (0, 0, 0) \neq (x, y, z)\}$

Notice that both of the above examples have a canonical ‘partial compactification’ where we erase the requirement that the coordinates are not zero. In fact, the map

$$G/U \rightarrow \overline{G/U} := \mathrm{Spec}(\mathrm{Fun}(G/U)) = \mathrm{Spec}(\mathrm{Fun}(G)^U)$$

turns out to be an open embedding: in other words, G/U is *quasi-affine*.

Theorem 0.18. The $\mathbb{C}$ -algebra $\mathrm{Fun}(G)^U$ is finitely generated.

It turns out that the variety $T^*(G/U)$ is also quasi-affine. Therefore, $T^*(G/U)$ is an open subset of its *affine closure* $\overline{T^*(G/U)} := \mathrm{Spec}(\mathrm{Fun}(T^*(G/U)))$. It is a remarkable fact (due to [DKS13] for $G = \mathrm{SL}_n$ and [GR15] in general) that $\mathrm{Fun}(T^*(G/U))$ is finitely generated as a $\mathbb{C}$ -algebra; this is in no way obvious!

Definition 0.19. The variety $\overline{T^*(G/U)}$ is the *holomorphic symplectic implosion* or, more briefly, the *universal implosion* for G .

Remark 0.20. When $G = \mathrm{SL}_n$, it is shown in [DKS13] that $\overline{T^*(\mathrm{SL}_n/U)}$ is, appropriately interpreted, hyperkähler. In this case, $\overline{T^*(\mathrm{SL}_n/U)}$ is sometimes referred to as the *universal hyperkähler implosion* for SL_n . However, such hyperkähler structure is not known for general G , and so we will stick to the terminology in Definition 0.19.

0.5. The Universal Implosion is a Coulomb Branch (w/ Williams). Recall that to any quiver Q with vertices labelled by a nonnegative integer n_v we can define a representation $\mathbf{N}_Q := \bigoplus_{v \rightarrow v'} \mathbb{A}^{n_v \times n_{v'}}$, which naturally is a representation of $\mathrm{GL}(Q) := \prod_v \mathrm{GL}_{n_v}$. It will become convenient (especially when giving this talk on a board) to simply write $\mathcal{M}_C(Q)$ instead of $\mathcal{M}_C(\mathrm{GL}(Q), \mathbf{N}_Q)$. We now have enough terminology to state one of my main results, joint with Harold Williams, which confirms a conjecture of Dancer-Hanany-Kirwan:

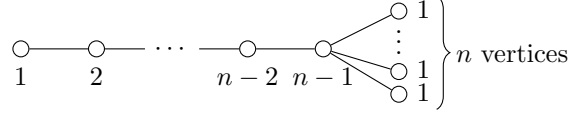
²We call a map $H \rightarrow G$ *gluable* for a representation $G \rightarrow \mathrm{GL}(\mathbf{N})$ if the representation has no pair of weights ξ_1, ξ_2 for a maximal torus T_G of G such that

- (1) $\xi_1|_{T_H} = \alpha \xi_2|_{T_H}$ for some $\alpha \in \mathbb{Q}$ and
- (2) for some cocharacter μ of T_G , the signs of the integers $\langle \xi_1 | \mu \rangle$ and $\langle \xi_2 | \mu \rangle$ are opposite, that is, $\langle \xi_1 | \mu \rangle \langle \xi_2 | \mu \rangle < 0$.

Theorem 0.21. [GW23] (see also [GW25]) There are isomorphisms

$$\overline{T^*(\mathrm{GL}_n/U)} \cong \mathcal{M}_C(Q), \text{ and } \overline{T^*(\mathrm{SL}_n/U)} \cong \mathcal{M}_C(\mathrm{GL}(Q)/\mathbb{G}_m, \mathbf{N}_Q)$$

where $Q_{\overline{1}}$ denotes the quiver



How can we use (6) to prove this? The idea is that if we restrict the A_n quiver to the group $\mathrm{GL}_1 \times \mathrm{GL}_2 \times \dots \times \mathrm{GL}_{n-1} \times \mathrm{GL}_1^n$, then we get the above quiver! The fact that $\overline{T^*(\mathrm{GL}_n/U)}$ actually admits the gluing description as in (6) can essentially be derived from [GK22] (who proved a very closely related result for semisimple G) and will follow directly from our discussion below.

0.6. Dismemberment and gluing of quiver gauge theories. Let Q denote a quiver with finite vertex set Q_0 and finite edge set Q_1 . We allow multiple edges incident to the same vertices and say that two edges in Q_1 are **parallel** if they join the same vertices, with the same or opposite orientations.

Definition 0.22. Given a quiver (Q_0, Q_1) , a *dismemberment* is a quiver $(\check{Q}_0, \check{Q}_1)$ equipped with a morphism of quivers $\gamma: (\check{Q}_0, \check{Q}_1) \rightarrow (Q_0, Q_1)$ which is a bijection between $\check{Q}_1 \rightarrow Q_1$ and a bijection on all vertices with no edges connected to them.

We can thus think of a dismemberment as keeping the edge set Q_1 fixed, but potentially carving each vertex in Q_0 into several separate vertices. Here is a concrete description of the glueable notion in the quiver case:

Corollary 0.23. If Q has no loops and all pairs of parallel edges lift to parallel edges in $\check{Q}$, there is an isomorphism

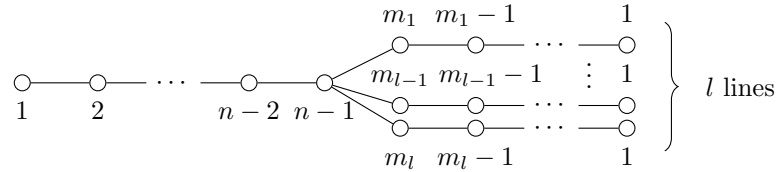
$$(7) \quad \mathcal{M}_C(Q, \mathbf{n}) \cong \mathcal{M}_C(\mathrm{GL}(Q, \mathbf{n}), 0) \times_{\mathrm{gl}(\check{Q}, \mathbf{n}) // \mathrm{GL}(\check{Q}, \mathbf{n})}^{\mathcal{M}_C(\mathrm{GL}(\check{Q}, \check{\mathbf{n}}), 0)} \mathcal{M}_C(\check{Q}, \check{\mathbf{n}})$$

of affine schemes.

This result says that Coulomb branches are quiver gauge theories are completely determined by Coulomb branches of quiver gauge theories of quivers which have exactly two vertices.

0.7. Corollaries for $\overline{T^*(\mathrm{GL}_n/U_P)}$. One specific instantiation of the above quiver results can be used to give a generalization of Theorem 0.21 to the case where U is replaced with the unipotent radical of a parabolic subgroup of GL_n or SL_n . Recall that any parabolic subgroup of GL_n or SL_n is conjugate to exactly one *standard parabolic subgroup*: a group of block upper triangular matrices in G with block sizes indexed by an ordered partition $n = m_1 + \dots + m_l$ of n , which we now fix once and for all and denote $\vec{m}$. In this case, it is not difficult to directly check that P is a semi-direct product of its unipotent radical U_P of block strictly upper triangular matrices with the *standard Levi subgroup* L of block diagonal matrices in G with block sizes indexed by $\vec{m}$.

For our fixed ordered partition $\vec{m}$, we let $Q_{\vec{m}}$ denote any quiver whose underlying graph is



where to each vertex, we label by a nonnegative integer. Our final main result states that $\overline{T^*(\mathrm{GL}_n/U_P)}$ is a Coulomb branch for the quiver gauge theory with quiver $Q_{\vec{m}}$:

Theorem 0.24. There is an isomorphism of affine varieties $\overline{T^*(\mathrm{GL}_n/U_P)} \cong \mathcal{M}_C(\mathrm{GL}_{Q_{\vec{m}}}, \mathbf{N}_{Q_{\vec{m}}})$.

The idea of this proof is that we can view the quiver representation from $Q_{\vec{m}}$ as obtained from a disjoint union of the type A_n quiver and the type $A_{\vec{m}}$ quiver, where $A_{\vec{m}}$ denotes any quiver whose underlying graph is

$$\begin{array}{ccccccc} m_1 & m_1 - 1 & & & 1 & & \\ \circ & \text{---} \circ & \text{---} & \cdots & \text{---} & \circ & \\ & & & & \vdots & & \\ m_{l-1} & m_{l-1} - 1 & & & 1 & & \\ \circ & \text{---} \circ & \text{---} & \cdots & \text{---} & \circ & \\ \circ & \text{---} \circ & \text{---} & \cdots & \text{---} & \circ & \\ m_l & m_l - 1 & & & 1 & & \end{array}$$

by pulling back by the map $\mathrm{GL}_{\vec{m}} \rightarrow \mathrm{GL}_n \times \mathrm{GL}_{\vec{m}}$.

0.7.1. Fun Corollaries. This implies that the ring of functions of $T^*(\mathrm{GL}_n/U_P)$ is finitely generated, which wasn't known. Also, using a result of [Bel23], one can deduce that $\overline{T^*(\mathrm{GL}_n/U_P)}$ has symplectic singularities, which is hard—it's not even known outside of type A . Also, it's possible to compare to quivers whose Coulomb branch descriptions were already known; for example, we have the following, suggested to us by Amihay Hanany:

Corollary 0.25. There is an isomorphism $\mathcal{N}_{\min} \cong \overline{T^*(\mathrm{SL}_4/U_{(2,2)})}$ of affine varieties, where $\mathcal{N}_{\min}$ is the closure of the minimal nonzero nilpotent orbit of $\mathfrak{e}_6$.

0.8. Derived geometric Satake and ring objects.

0.8.1. Digression on Algebra Objects. Observe that, given any monoidal category $\mathcal{A}$, we can talk about the *algebra objects* in this category—these are given by an object $A \in \mathcal{A}$ along with a ‘multiplication’ map $A \star A \rightarrow A$ and a unit map $\mathbf{1}_{\mathcal{A}} \rightarrow A$ such that (informally) multiplication is associative and multiplication by the unit is the identity. Similarly, if the monoidal structure is braided, we can speak of *commutative algebra objects* in our braided monoidal category.

Example 0.26. Assume B is a commutative ring. Then the category $B\text{-mod}$ admits a monoidal structure by $- \otimes_B -$. An algebra object in $B\text{-mod}$ is the data of an B -module A , a multiplication map $A \otimes_B A \rightarrow A$ of B -modules, and a unit map $B \rightarrow A$. It's an elementary exercise to construct a bijection

$$\{\text{commutative algebra objects } A \in B\text{-mod}\} \xrightarrow{\sim} \{\text{affine schemes } \mathrm{Spec}(A) \text{ over } \mathrm{Spec}(B)\}$$

so we can geometrically view commutative algebra objects $A \in B\text{-mod}$ as ‘living over’ $\mathrm{Spec}(B)$.

0.8.2. Coulomb Branches Via Ring Objects. The convolution structure on $H^*(\mathrm{Gr}_G)$ in fact can be viewed as a corollary of the fact that the category of constructible sheaves $\mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ admits a *convolution monoidal structure*—informally, given any $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ we can define their ‘convolution’ $\mathcal{F} \star \mathcal{G} \in \mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ with associativity and unit isomorphisms.

Proposition 0.27. [BFN19] To any $(G, \mathbf{N})$,

$$\mathcal{A}_{G, \mathbf{N}} := \pi_*(\omega_{\mathcal{R}_{G, \mathbf{N}}}) \in \mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$$

naturally upgrades to the structure of an algebra object.

It's an elementary argument that the equivariant cohomology functor

$$H^*_{G(\mathbb{C}[[t]])}(-) : (\mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G), *) \rightarrow (H^*_{G_{\mathcal{O}}}(\mathrm{pt})\text{-mod}, - \otimes_{H^*_{G_{\mathcal{O}}}(\mathrm{pt})} -)$$

is monoidal. Since monoidal functors send algebra objects to algebra objects, we get:

Corollary 0.28. The algebra of functions $H^*_{G(\mathbb{C}[[t]])}(\mathcal{A}_{G, \mathbf{N}})$ on the Coulomb branch is an algebra object of $H^*_{G_{\mathcal{O}}}(\mathrm{pt})\text{-mod}$. Equivalently, by Example 0.26, $\mathcal{M}_C(G, \mathbf{N})$ lies over $\mathrm{Spec}(H^*_{G(\mathbb{C}[[t]])}(\mathrm{pt})\text{-mod})$.

It's also standard that

$$\mathrm{Spec}(H^*_{G(\mathbb{C}[[t]])}(\mathrm{pt})) \cong \mathbb{A}^{\mathrm{rank}(G)} \cong \mathfrak{t} // W \xrightarrow{\sim} \mathfrak{g} // G := \mathrm{Spec}(\mathcal{O}(\mathfrak{g})^G)$$

so $\mathcal{M}_C(G, \mathbf{N})$ lies over some affine space.

0.8.3. *Derived Geometric Satake.* An advantage of the above perspective is that $\mathcal{A}_{G,\mathbf{N}}$ is an interesting object and a priori contains more information than the Coulomb branch itself. To unpack this, recall the *derived geometric Satake equivalence*:

Theorem 0.29. [BF08] There is an equivalence of categories $\mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G) \cong \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(G^\vee))$.

Thus algebra objects in $\mathrm{Shv}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G)$ give us algebra objects in $\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(G^\vee))$. At first pass, one can view the cohomological shift in $\mathrm{Sym}(\check{\mathfrak{g}}[-2])$ as a grading. Commutative algebra objects in the abelian category $\mathrm{Sym}(\check{\mathfrak{g}}(-2))\text{-mod}(\mathrm{Rep}(G^\vee))$ (where we replace the cohomological shift with a grading) can be shown to be equivalent to the category of affine schemes $\mathrm{Spec}(B)$ equipped with a $G \times \mathbb{G}_m$ -action and an equivariant map $\mathrm{Spec}(B) \rightarrow \check{\mathfrak{g}}^*$. Following [BZSV24] (see also [Nak24]), we can make the following definition:

Definition 0.30. For a representation $\mathbf{N}$ of a complex reductive group G , its *S-dual* is the affine scheme obtained from the above procedure.

We conjectured that it's possible to recover the *S-dual* from the Coulomb branch itself and its residual symmetries. This was recently announced by Ginzburg:

Theorem 0.31. For any representation $\mathbf{N}$ of a complex reductive group G , the *S-dual* of G is the affine scheme $(G \times \mathfrak{g} // G) \times_{\mathfrak{g} // G}^{\mathcal{M}_C(G,0)} \mathcal{M}_C(G, \mathbf{N})$.

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